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Factorizations of Normalized Totally Positive Systems

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Abstract. The de Casteljau algorithm for evaluation of Bézier curves can be generalized to curves generated by any normalized totally positive basis. The construction of this algorithm is based upon a factorization of the system as a product of bidiagonal stochastic matrices of functions. These factorizations depend on a selection of a sequence of rectangular bidiagonal matrices of decreasing dimensions.

§1. Introduction

The Bernstein basis $b_i^n(t) := \binom{n}{i}(1-t)^{n-i}t^i$ can be used for defining a Bézier curve

$$\gamma(t) := \sum_{i=0}^{n} P_i b_i^n(t), \quad t \in [0,1].$$

By means of the degree raising technique, we can express the Bézier curve in terms of the Bernstein basis of one higher degree: $\gamma(t) = \sum_{i=0}^{n+1} Q_i b_i^{n+1}(t)$, $t \in [0,1]$. Indeed, the relations

$$b_i^n(t) = \frac{n-i+1}{n+1} b_i^{n+1}(t) + \frac{i+1}{n+1} b_{i+1}^{n+1}(t), \quad i = 0,\ldots,n, \tag{1.1}$$

can be written in matrix form as

$$(b_0^n, \ldots, b_n^n) = (b_0^{n+1}, \ldots, b_{n+1}^{n+1})A, \tag{1.2}$$

where A is an $(n+2)\times(n+1)$ nonnegative stochastic bidiagonal matrix. Such a matrix can be written as:

$$A = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ \alpha_1 & 1-\alpha_1 & 0 & \cdots & 0 \\ 0 & \alpha_2 & 1-\alpha_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \alpha_n & 1-\alpha_n \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}, \quad 0 \le \alpha_i \le 1, \qquad i = 1,\ldots,n. \tag{1.3}$$

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Equality (1.2) corresponds to the choice

$$\alpha_i := \frac{i}{n+1}, \quad i = 1, \dots, n. \tag{1.4}$$

Using (1.2), we can write

$$\gamma(t) = (b_0^n, \dots, b_n^n)(P_0, \dots, P_n)^T = (b_0^{n+1}, \dots, b_{n+1}^{n+1})A(P_0, \dots, P_n)^T,$$

which proves that the new control polygon is given by

$$(Q_0, \dots, Q_{n+1})^T := A(P_0, \dots, P_n)^T. \tag{1.5}$$

On the other hand, the de Casteljau algorithm for the pointwise evaluation of the curve is based on the following well-known recurrence relations

$$b_i^{n+1}(t) = \lambda_{i-1}(t)b_{i-1}^n(t) + (1 - \lambda_i(t))b_i^n(t), \quad i = 0, \dots, n+1, \tag{1.6}$$

where $\lambda_i(t) := t$ for $i = 0, \dots, n$, $\lambda_{-1}(t) := 0$, and $\lambda_{n+1}(t) := 1$. Indeed, we can write (1.6) as

$$(b_0^{n+1}(t), \dots, b_{n+1}^{n+1}(t)) = (b_0^n(t), \dots, b_n^n(t))\Lambda(t), \tag{1.7}$$

where $\Lambda(t)$ denotes the nonnegative stochastic bidiagonal matrix

$$\Lambda(t) = \begin{pmatrix} 1-\lambda_0(t) & \lambda_0(t) & & \\ & \ddots & \ddots & \\ & & 1-\lambda_n(t) & \lambda_n(t) \end{pmatrix}. \tag{1.8}$$

Then, starting with a Bézier curve $\gamma(t) = \sum_{i=0}^{n+1} Q_i b_i^{n+1}(t)$, (1.7) gives

$$\gamma(t) = (b_0^{n+1}, \dots, b_{n+1}^{n+1})(Q_0, \dots, Q_{n+1})^T = (b_0^n, \dots, b_n^n)(P_0(t), \dots, P_n(t))^T,$$

where

$$(P_0(t), \dots, P_n(t))^T := \Lambda(t)(Q_0, \dots, Q_{n+1})^T. \tag{1.9}$$

Equality (1.9) describes the first step of the de Casteljau algorithm for the evaluation of $\gamma(t)$.

Bernstein bases are totally positive on $[0,1]$. In this paper we shall prove that similar properties hold for any totally positive basis of functions. Let us recall that a totally positive matrix is a matrix such that all of its minors are nonnegative. A totally positive system of functions defined on I is a system $(u_0, \dots, u_n)$ such that $(u_j(t_i))_{0\le i,j\le n}$ is totally positive for all $t_0 < \cdots < t_n$ in I. A normalized totally positive (NTP) basis $(u_0, \dots, u_n)$ is a totally positive system of linearly independent functions such that $\sum_{i=0}^n u_i = 1$.

Given an NTP basis $(u_0^{n+1}, \dots, u_{n+1}^{n+1})$ on an interval I, and a nonnegative stochastic $(n+2)\times(n+1)$ matrix A of rank $n+1$, we shall consider the system of functions defined by

$$(u_0^n, \dots, u_n^n) := (u_0^{n+1}, \dots, u_{n+1}^{n+1})A. \tag{1.10}$$

Starting from a curve $\gamma(t) = \sum_{i=0}^{n} P_i u_i^n(t)$, $t \in I$, clearly (1.10) allows us to express it as $\gamma(t) = \sum_{i=0}^{n+1} Q_i u_i^{n+1}(t)$, where the points Q_i are defined by (1.5). In this paper we will derive from (1.10) the existence of nondecreasing functions $\lambda_0, \ldots, \lambda_n$ with values in $[0,1]$ such that

$$(u_0^{n+1}(t), \ldots, u_{n+1}^{n+1}(t)) = (u_0^n(t), \ldots, u_n^n(t))\Lambda(t), \quad t \in I. \tag{1.11}$$

The matrix $\Lambda(t)$ is defined from λ_i as in (1.8). Starting with a curve $\gamma(t) = \sum_{i=0}^{n+1} Q_i u_i^{n+1}(t)$, $t \in I$, we will be able to write it as $\gamma(t) = \sum_{i=0}^{n} P_i(t) u_i^n(t)$, where the points $P_0(t), \ldots, P_n(t)$ are again given by (1.9). On the other hand, we shall check that (1.10) implies that $(u_0^n, \ldots, u_n^n)$ is an NTP basis on I. It will therefore be possible to iterate this process. Doing so, we shall obtain a de Casteljau type algorithm for the evaluation of $\gamma(t)$.

Pottmann and Mazure in [5,6,7] developed generalizations (1.11) of the de Casteljau algorithm for Tchebycheffian curves. Here we show that these generalizations can be also obtained for any curve generated by an NTP basis.

We observe that for each value of t, the point $P_i^j(t)$ is a convex combination of two consecutive points, obtained in the previous step of the algorithm. Therefore, these algorithms can be seen as corner cutting algorithms for curve evaluation [4].

§2. Recurrence Relations for NTP Systems

The following proposition allows us to describe the generalization of formulae (1.6) to any NTP systems related by a matrix (1.3). First we need to show the following auxiliary result.

Lemma 2.1. *Let $(u_0^{n+1}, \ldots, u_{n+1}^{n+1})$ be an NTP basis of functions defined on I and*

$$(u_0^n, \ldots, u_n^n) := (u_0^{n+1}, \ldots, u_{n+1}^{n+1})A, \tag{2.1}$$

where $A \in \mathbb{R}^{(n+2)\times(n+1)}$ is of the form (1.3). Let $C_i := \{t \in I \mid u_i^n(t) \neq 0\}$. Then

(i) *$u_{i+1}^{n+1}(t)/u_i^n(t)$, $t \in C_i$, is a nondecreasing function,*
(ii) *$\alpha_{i+1} u_{i+1}^{n+1}(t)/u_i^n(t) \in [0,1]$, for all $t \in C_i$.*

Proof:

(i) Since A is bidiagonal, we can write

$$u_i^n(t) = (1-\alpha_i)u_i^{n+1}(t) + \alpha_{i+1}u_{i+1}^{n+1}(t), \quad t \in I, \qquad i = 0, \ldots, n. \tag{2.2}$$

Observe that, since A is nonnegative and $(u_0^{n+1}, \ldots, u_{n+1}^{n+1})$ is NTP, then $u_i^n(t) > 0$, for all $t \in C_i$. Moreover,

$$\begin{vmatrix} u_i^n(t) & u_{i+1}^{n+1}(t) \\ u_i^n(s) & u_{i+1}^{n+1}(s) \end{vmatrix} = (1-\alpha_i) \begin{vmatrix} u_i^{n+1}(t) & u_{i+1}^{n+1}(t) \\ u_i^{n+1}(s) & u_{i+1}^{n+1}(s) \end{vmatrix} \geq 0, \quad t < s, \tag{2.3}$$

because $(u_0^{n+1}, \ldots, u_{n+1}^{n+1})$ is totally positive. Formula (2.3) implies that u_{i+1}^{n+1}/u_i^n is nondecreasing in C_i.

(ii) Using (2.2), we can write

$$0 \le \frac{\alpha_{i+1} u_{i+1}^{n+1}(t)}{u_i^n(t)} = \frac{\alpha_{i+1} u_{i+1}^{n+1}(t)}{(1-\alpha_i) u_i^{n+1}(t) + \alpha_{i+1} u_{i+1}^{n+1}(t)} \le 1, \quad t \in C_i. \quad \square$$

The following proposition is devoted to showing that formula (1.11) holds for NTP bases.

Proposition 2.2. *Let $(u_0^{n+1}, \dots, u_{n+1}^{n+1})$ and $(u_0^n, \dots, u_n^n)$ be two NTP bases of functions on I related by (2.1), where A is a matrix (1.3) ($\operatorname{rank} A = n+1$). Let $C_i := \{t \in I \mid u_i^n(t) \ne 0\}$. Then the functions $\lambda_i : I \to \mathbb{R}$, $i = 0, \dots, n$, defined by*

$$\lambda_i(t) := \begin{cases} \alpha_{i+1} \inf \left\{ u_{i+1}^{n+1}(s)/u_i^n(s) \mid s \in C_i \right\}, & \text{if } u_i^n(s) = 0,\ \forall s \le t, \\ \alpha_{i+1} \sup \left\{ u_{i+1}^{n+1}(s)/u_i^n(s) \mid s \in C_i, s \le t \right\}, & \text{otherwise,} \end{cases} \tag{2.4}$$

are nondecreasing, and satisfy

$$0 \le \lambda_i(t) \le 1, \quad \forall t \in I, \qquad i = 0, \dots, n. \tag{2.5}$$

Furthermore, if we use definition (1.8), then (1.11) holds.

Proof: Since $(u_0^n, \dots, u_n^n)$ are linearly independent, then $C_i \ne \emptyset$ for all i. Therefore, by Lemma 2.1 (ii), we can define

$$\kappa_i := \alpha_{i+1} \inf \left\{ \frac{u_{i+1}^{n+1}(s)}{u_i^n(s)} \,\middle|\, s \in C_i \right\} \in [0, 1].$$

If the condition $u_i^n(s) = 0$, $\forall s \le t$, does not hold, then the set $\{s \in C_i \mid s \le t\}$ is nonempty and by Lemma 2.1 (ii), we can define

$$\kappa_i \le \lambda_i(t) := \alpha_{i+1} \sup \left\{ \frac{u_{i+1}^{n+1}(s)}{u_i^n(s)} \,\middle|\, s \in C_i, s \le t \right\} \le 1.$$

We have seen that λ_i, $i = 0, \dots, n$, are well-defined and that (2.5) holds. In order to see that $\lambda_i(t)$ are nondecreasing, let us observe first that if $\{s \in C_i \mid s \le t_1\} = \emptyset$ and $\{s \in C_i \mid s \le t_2\} \ne \emptyset$, then t_1 must be less than t_2. Therefore, we only have to show that $\lambda_i(t_1) \le \lambda_i(t_2)$ only for all $t_1 < t_2$ such that there exists some $s < t_1$ with $u_i(s) \ne 0$. We observe that $\{s \in C_i \mid s \le t_1\} \subseteq \{s \in C_i \mid s \le t_2\}$. Therefore, by Lemma 2.1 (i),

$$\alpha_{i+1} \sup \left\{ \frac{u_{i+1}^{n+1}(s)}{u_i^n(s)} \,\middle|\, s \in C_i, s \le t_1 \right\} \le \alpha_{i+1} \sup \left\{ \frac{u_{i+1}^{n+1}(s)}{u_i^n(s)} \,\middle|\, s \in C_i, s \le t_2 \right\}.$$

We now establish the relation

$$\alpha_{i+1} u_{i+1}^{n+1}(t) = \lambda_i(t) u_i^n(t), \quad \forall t \in I, \quad i = 0, \dots, n. \tag{2.6}$$

If $u_i^n(t) = 0$, then by (2.2), $\alpha_{i+1} u_{i+1}^{n+1}(t) = 0$, and (2.6) trivially holds. Otherwise, we have $t \in \{s \in C_i \mid s \le t\}$, and by Lemma 2.1 (i),

$$\lambda_i(t) = \alpha_{i+1} u_{i+1}^{n+1}(t)/u_i^n(t),$$

so (2.6) is confirmed again.

Finally, using (2.6) and (2.2) we can write

$$\lambda_{i-1}(t) u_{i-1}^n(t) + (1 - \lambda_i(t)) u_i^n(t) = \alpha_i u_i^{n+1}(t) + u_i^n(t) - \alpha_{i+1} u_{i+1}^{n+1}(t) = u_i^{n+1}(t)$$

for all $t \in I$ and $i = 0, \dots, n$. $\square$

§3. A Generalization of the de Casteljau Algorithm for NTP Bases

Given an NTP basis $(u_0^n,\dots,u_n^n)$ of a space $\mathcal{U}^n$ of functions defined on I, we can obtain a sequence of NTP bases $(u_0^k,\dots,u_k^k)$ of $(k+1)$-dimensional subspaces $\mathcal{U}^k$ by the recurrence

$$(u_0^k(t),\dots,u_k^k(t)) := (u_0^{k+1}(t),\dots,u_{k+1}^{k+1}(t))A_{k+1},\ k=n-1,n-2,\dots,0, \tag{3.1}$$

where $A_{k+1}\in\mathbb{R}^{(k+2)\times(k+1)}$ is a matrix of type (1.3), $\operatorname{rank} A_{k+1}=k+1$.

In fact, since A_{k+1} are nonnegative bidiagonal matrices, it easily follows, using Theorem 2.3 of [1], that A_{k+1} is totally positive and, using the Cauchy-Binet formula (formula (1.23) of [1]), that the systems (3.1) are totally positive. Taking into account that $(u_0^{k+1},\dots,u_{k+1}^{k+1})$ is normalized and A_{k+1} is stochastic, we derive that the systems (3.1) are also normalized. Furthermore, formula (3.1) relates two bases if and only if $\operatorname{rank} A_{k+1}=k+1$. Observe that $\operatorname{rank} A_{k+1}<k+1$ if and only if there exist $1\le i<j\le k$ such that $\alpha_i^{k+1}=1$ and $\alpha_j^{k+1}=0$.

Let us observe that the subspaces $\mathcal{U}^k$ form a chain, that is,

$$\mathcal{U}^n\supset\mathcal{U}^{n-1}\supset\cdots\supset\mathcal{U}^1\supset\mathcal{U}^0=\operatorname{span}\{1\}.$$

Moreover, since (u_0^0) is an NTP basis of $\mathcal{U}^0$ then $u_0^0(t)=1$, for all $t\in I$.

By Proposition 2.2, the bases of (3.1) are related by

$$(u_0^{k+1}(t),\dots,u_{k+1}^{k+1}(t))=(u_0^k(t),\dots,u_k^k(t))\Lambda_{k+1}(t),\quad t\in I, \tag{3.2}$$

where $\Lambda_{k+1}(t)$ is a matrix of type (1.8). We shall denote by $\lambda_i^{k+1}(t)$ the $(i+1,i+2)$ entry of $\Lambda_{k+1}(t)$. The recurrences (3.1) and (3.2) give

$$(u_0^k(t),\dots,u_k^k(t))=(u_0^n(t),\dots,u_n^n(t))A_n\cdots A_{k+2}A_{k+1},\quad t\in I, \tag{3.3}$$

and

$$(u_0^k(t),\dots,u_k^k(t))=\Lambda_1(t)\Lambda_2(t)\cdots\Lambda_k(t),\quad t\in I, \tag{3.4}$$

for $k=0,\dots,n$, with the convention $A_n\cdots A_{k+1}$ equals the identity matrix when $k=n$ and $\Lambda_1(t)\cdots\Lambda_k(t)$ equals the scalar constant 1 when $k=0$.

Formulae (3.4) can be interpreted as a factorization of the NTP system $(u_0^k,\dots,u_k^k)$ as a product of bidiagonal stochastic matrices of functions.

Let us summarize all the conclusions in the following theorem.

Theorem 3.1. *Let $(u_0^n,\dots,u_n^n)$ be an NTP basis of functions defined on I. Let $A_k\in\mathbb{R}^{(k+1)\times k}$, $k=1,\dots,n$, be matrices (3.1) of maximal rank. Define NTP systems $(u_0^k,\dots,u_k^k)$, $k=0,\dots,n-1$, by (3.1) (or equivalently by (3.3)). Then there exist matrices $\Lambda_k(t)$ of type (1.8) whose $(i+1,i+2)$ entry $\lambda_i^k(t)$ is nondecreasing on I and with values in $[0,1]$, $k=1,\dots,n$, such that (3.2) and (3.4) hold. In particular,*

$$(u_0^n(t),\dots,u_n^n(t))=\Lambda_1(t)\cdots\Lambda_n(t),\quad \forall t\in I. \tag{3.5}$$

Moreover, for any control polygon $P_0 \cdots P_n$ consider the following generalization of the de Casteljau algorithm:

$$
\begin{aligned}
&\textbf{for } j = 0, 1, \ldots, n \\
&\qquad P_j^n(t) := P_j \\
&\textbf{for } i = n-1, \ldots, 1, 0 \\
&\qquad \textbf{for } j = 0, 1, \ldots, i \\
&\qquad\qquad P_j^i(t) := (1 - \lambda_j^{i+1}(t)) P_j^{i+1}(t) + \lambda_j^{i+1}(t) P_{j+1}^{i+1}(t)
\end{aligned}
$$

At each step we have

$$\gamma(t) = \sum_{j=0}^{i} P_j^i(t) u_j^i(t), \quad t \in I, \qquad i = 0, \ldots, n. \tag{3.6}$$

In particular $\gamma(t) = P_0^0(t)$ for all $t \in I$, that is, this generalized de Casteljau algorithm reconstructs the curve from its control polygon.

Proof: The existence of the matrices $\Lambda_k(t)$ of type (1.8), satisfying (3.1) and (3.2) follows from Proposition 2.2. From the algorithm we see that

$$(P_0^i(t), \ldots, P_i^i(t))^T = \Lambda_{i+1}(t)(P_0^{i+1}(t), \ldots, P_{i+1}^{i+1}(t))^T,$$

and by (3.5) we can write

$$
\begin{aligned}
\gamma(t) =&(u_0^n(t), \ldots, u_n^n(t))(P_0, \ldots, P_n)^T = \\
&\Lambda_1(t) \cdots \Lambda_k(t) \Lambda_{k+1}(t) \cdots \Lambda_n(t)(P_0, \ldots, P_n)^T = \\
&(u_0^k(t), \ldots, u_k^k(t))(P_0^k(t), \ldots, P_k^k(t))^T. \quad \square
\end{aligned}
$$

Example 3.2. When applying Proposition 2.2 to (1.1) or (1.2), the functions that we obtain are $\lambda_i(t) = t$, $i = 0, \ldots, n$. Hence we obtain (1.7), and the corresponding algorithm described in Theorem 3.1 is just the classical de Casteljau algorithm for polynomials. Of course, any other choice of a sequence $(A_k), k = 1, \ldots, n$, of nonnegative stochastic matrices of maximal rank could lead to another de Casteljau type algorithm. For instance, if we consider the Bernstein basis (b_0^2, b_1^2, b_2^2) of degree 2, the matrix

$$A_2 := \begin{pmatrix} 1 & 0 \\ 1/3 & 2/3 \\ 0 & 1 \end{pmatrix}$$

defines a NTP basis $((1-t)(3-t)/3, t(4-t)/3)$ on $[0,1]$. This system generates a subspace of quadratic functions, different from the polynomials of degree less than or equal to 1. Furthermore, the functional matrices obtained by applying Proposition 2.2

$$\Lambda_1(t) = \begin{pmatrix} \frac{(1-t)(3-t)}{3} & \frac{t(4-t)}{3} \end{pmatrix}, \quad \Lambda_2(t) = \begin{pmatrix} \frac{3(1-t)}{3-t} & \frac{2t}{3-t} & 0 \\ 0 & \frac{4(1-t)}{4-t} & \frac{3t}{4-t} \end{pmatrix},$$

lead to a corner cutting algorithm different from the classical de Casteljau algorithm.

In [2], it was shown that, in any space with an NTP basis, there exists a particular NTP basis called the normalized B-basis which has the optimal shape preserving properties among all NTP bases of the space. In Theorem 4.3 of [3] it was shown that if $(u_0^{n+1}, \dots, u_{n+1}^{n+1})$ is a normalized B-basis of an $(n+2)$-dimensional space and $(u_0^n, \dots, u_n^n)$ is a B-basis of an $(n+1)$-dimensional subspace, then there exists a matrix A (1.3) such that

$$(u_0^n, \dots, u_n^n) = (u_0^{n+1}, \dots, u_{n+1}^{n+1})A.$$

Thus, B-bases provide good examples of when Theorem 3.1 can be applied. In the case of polynomial spline spaces (see [2]), the normalized B-basis is precisely the B-spline basis.

Let $\mathcal{T} = \{t_0 = \cdots = t_{k-1} < t_k \leq \cdots \leq t_n < t_{n+1} = \cdots = t_{n+k}\}$, $t_i < t_{i+k}$, for all i, be an extended knot sequence and

$$N_{i,\mathcal{T}}^k(t) := (t_{i+k} - t_i)[t_i, \dots, t_{i+k}](\cdot - t)_+^{k-1}, \quad t \in [t_0, t_{n+1}],\ i = 0, \dots, n,$$

the associated B-spline basis of the space $\mathcal{S}_\mathcal{T}^k$. Let us insert a knot τ in $\mathcal{T}$ such that $t_j \leq \tau < t_{j+1}$ (if $\tau = t_j$ then the multiplicity of t_j must be less than k) and define a new sequence of knots $\hat{\mathcal{T}}$

$$\hat{t}_i := \begin{cases} t_i, & 0 \leq i \leq j, \\ \tau, & i = j+1, \\ t_{i-1}, & j+2 \leq i \leq n+k+1. \end{cases}$$

The normalized B-bases of $\mathcal{S}_\mathcal{T}^k$ and $\mathcal{S}_{\hat{\mathcal{T}}}^k$ are related by a matrix (1.3) with

$$\alpha_i := \begin{cases} 0, & 0 \leq i \leq j-k+1, \\ (t_{i+k-1} - \tau)/(t_{i+k-1} - t_i), & j-k+2 \leq i \leq j, \\ 1, & j+1 \leq i \leq n. \end{cases}$$

Applying Proposition 2.2 to both B-spline bases, a relation (1.11) is obtained. In order to obtain a generalized de Casteljau algorithm, we first remove succesively all interior knots until we arrive at the Bernstein basis. Then we can continue with the steps of an evaluation algorithm for polynomials (e.g., the de Casteljau algorithm). We illustrate this procedure with a simple example:

Example 3.3. Take $\hat{\mathcal{T}} := (0,0,1/2,1,1)$, $\mathcal{T} := (0,0,1,1)$. The associated B-spline bases are related by a matrix (1.3):

$$\left(N_{0,\mathcal{T}}^2(t), N_{1,\mathcal{T}}^2(t)\right) = \left(N_{0,\hat{\mathcal{T}}}^2(t), N_{1,\hat{\mathcal{T}}}^2(t), N_{2,\hat{\mathcal{T}}}^2(t)\right) \begin{pmatrix} 1 & 0 \\ 1/2 & 1/2 \\ 0 & 1 \end{pmatrix}, \quad t \in [0,1].$$

Using Proposition 2.2, we obtain that these bases are also related by (1.11) with a matrix (1.8), where

$$\lambda_0(t) = \min(1, t/(1-t)), \quad \lambda_1(t) = \max(0, (2t-1)/t), \quad t \in [0,1].$$

The evaluation algorithm for $\gamma(t) := \sum_{i=0}^{2} P_i N_{i,\hat{T}}^2(t)$ can be described as follows. First compute

$$P_0^1(t) := (1-\lambda_0(t))P_0 + \lambda_0(t)P_1, \quad P_1^1(t) := (1-\lambda_1(t))P_1 + \lambda_1(t)P_2,$$

and then $\gamma(t) = (1-t)P_0^1(t) + tP_1^1(t)$. Note that the last step of the algorithm corresponds to the de Casteljau algorithm. Of course, this algorithm is different from the classical de Boor-Cox algorithm for evaluation of B-splines.

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